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BOUNDED DOMAINS OF THE FATOU SET OF AN ENTIRE FUNCTION

BY

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ABSTRACT

Let f be an entire function of positive lower order and order less than 1/2. It is shown that every component of its Fatou set is bounded.

1. Introduction

Let f be a nonlinear entire function of the complex variable z. Its natural iterates f^n are defined by $f^0(z) = z$, $f^1(z) = f(z)$, $f^{n+1} = f(f^n(z))$, $n = 1, 2, \ldots$. The **Fatou set** $\mathcal{F}(f)$ of the function f is the largest open set of the complex plane where the family $\{f^n\}$ forms a normal family. The complement of $\mathcal{F}(f)$ is called the **Julia set** and denoted by $\mathcal{J}(f)$. Then $\mathcal{J}(f)$ is closed and completely invariant under f. For more details of the concepts and properties in the iteration theory, we refer to the books by Beardon [5], Carleson and Gamelin [7] and McMullen [10] as well as Milnor's [9] lecture notes for rational functions and the survey articles of Baker [4] and Eremenko and Lyubich [8] for rational and entire functions and Bergweiler [6] for transcendental meromorphic functions.

If f is a polynomial of degree at least two, then $\mathcal{F}(f)$ contains the component $D = \{z, f^n(z) \to \infty\}$, which is unbounded and completely invariant. If f is transcendental entire, then, from Picard's theorem and the invariance of $\mathcal{J}(f)$, it is clear that $\mathcal{J}(f)$ is unbounded, so that $\mathcal{F}(f)$ no longer contains a neighbourhood of ∞.

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Baker [3] raised the question of whether every component of $\mathcal{F}(f)$ must be bounded if f is of sufficiently small growth. The appropriate growth condition would be of order $< 1/2$, since Baker [3] showed that for any sufficiently large positive a, the function $f_0(z) = z^{-1/2}\sin z^{1/2} + z + a$ is of order $\rho = 1/2$ and has an unbounded component D of $\mathcal{F}(f)$ containing a segment $[x_0, \infty)$ of the positive real axis. Moreover, Baker proved that the growth of a transcendental entire function f must exceed order $1/2$, minimal type, if $\mathcal{F}(f)$ has an unbounded invariant component. In the positive direction to this problem, a few results have been obtained. Baker [3] proved that if a function of order $\rho = 0$ with sufficiently small growth, then $\mathcal{F}(f)$ has no unbounded components.

THEOREM A: *Let f be an entire function with*

$$\log M(r,f) = O\{(\log r)^t\} \tag{1.1}$$

as $r \to \infty$, where $1 < t < 3$. Then every component of $\mathcal{F}(f)$ is bounded.

Here and later, we use the standard notations for the maximum modulus $M(r,f)$, minimum modulus $L(r,f)$, order of growth ρ and lower order of growth μ of a function f, namely,

$$M(r,f) = \max\{|f(z)| : |z| = r\},$$

$$L(r,f) = \min\{|f(z)| : |z| = r\},$$

$$\rho = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r},$$

and

$$\mu = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

Stallard [11] improved the sufficient condition (1.1) and obtained

THEOREM B: *Let f be an entire function with*

$$\log\log M(r,f) = O\left(\frac{(\log r)^{1/2}}{(\log\log r)^{\varepsilon}}\right) \tag{1.2}$$

as $r \to \infty$, where $\varepsilon \in (0,1)$. Then every component of $\mathcal{F}(f)$ is bounded.

We note that the entire functions satisfying (1.2) are still of order $\rho = 0$ with rather small growth. By imposing a condition on the regularity of growth, Stallard also proved the following

THEOREM C: *Let f be an entire function of order $\rho < \frac{1}{2}$ such that*

$$\frac{\log M(2r, f)}{\log M(r, f)} \to c, \quad \text{as } r \to \infty, \tag{1.3}$$

where c is a finite constant that depends only on f. Then every component of $\mathcal{F}(f)$ is bounded.

By a method which is somewhat different from those of Baker or Stallard, Anderson and Hinkkanen [1] obtained the following result under another regularity condition on the growth of f.

THEOREM D: *Let f be an entire function of order $\rho < \frac{1}{2}$ such that, for some positive constant c,*

$$\frac{\phi'(x)}{\phi(x)} \geq \frac{1+c}{x} \tag{1.4}$$

for all sufficiently large x, where $\phi(x) = \log M(e^x, f)$. Then every component of $\mathcal{F}(f)$ is bounded.

We note that $\phi(x)$ is an increasing convex function of x by the Hadamard three-circles theorem and the function $\phi'(x)$ may fail to exist at a countable set of points. At such points, $\phi'(x)$ is defined to be the right-hand derivative.

In this paper we give a positive answer to this problem for all functions of positive lower order.

THEOREM 1: *Let f be an entire function of order less than 1/2. Then if its lower order $\mu > 0$, every component of $\mathcal{F}(f)$ is bounded.*

Therefore only the case $\mu = 0$ remains open.

2. Lemmas and proof of the theorems

To prove our main theorem, we need the following results.

The $\cos \pi\rho$-type theorem plays a fundamental role in our proof. The key point is that for an entire function f of order less than $\frac{1}{2}$, the $\cos \pi\rho$-type theorem ensures that the iterates f^k have a certain property of self-sustaining spread on any compact subset of the components of the Fatou set of f. The most suitable form for us is the following lemma, due to Baker [2].

LEMMA 1: *Let f be an entire function of order $\rho < \frac{1}{2}$. Then there exist $m > 1$, $R > 0$ such that, for all $r > R$, there exists r' satisfying*

$$r \leq r' \leq r^m \quad \text{and} \quad L(r', f) = M(r, f).$$

The next result was also obtained by Baker [3], using Schottky's theorem.

LEMMA 2: *If, in a domain D, the analytic functions g of the family G omit the values $0, 1$, and if E is a compact subset of D on which the functions all satisfy $|g(z)| \geq 1$, then there exist constants B, C dependent only on E and D, such that for any z, z' in E and any g in G we have*

$$|g(z')| < B|g(z)|^C.$$

We shall show the following result, which implies Theorem 1.

THEOREM 2: *Let f be an entire function of order $< \frac{1}{2}$. If for some positive number R_1, and $R_{n+1} = M(R_n, f)$ $(n = 1, 2, \ldots)$, there exists a number $T > 1$ such that*

$$M(R_n^T, f) > M(R_n, f)^{mT}, \tag{2.1}$$

for all sufficiently large n, where m is the number in Lemma 1, then the Fatou set $\mathcal{F}(f)$ has no unbounded components.

Proof of Theorem 2: We suppose on the contrary that $\mathcal{F}(f)$ has an unbounded component D. Without loss of generality we may assume that $0, 1$ belong to $\mathcal{J}(f)$. Hence each function f^k omits the value $0, 1$ in D. It follows from (2.1) that there exists $N_0 \in \mathbb{N}$ such that

$$M({R_n}^T, f) > M(R_n, f)^{mT} \tag{2.2}$$

for all $n \geq N_0$. From Lemma 1, we can choose N_1 $(\geq N_0)$ such that for each $n \geq N_1$, there exists ρ_n satisfying

$$(R_n)^T \leq \rho_n \leq (R_n)^{mT} \tag{2.3}$$

with

$$L(\rho_n, f) = M({R_n}^T, f). \tag{2.4}$$

Hence (2.2) and (2.4) give

$$L(\rho_n, f) > (R_{n+1})^{mT}. \tag{2.5}$$

On the other hand, since D is unbounded and connected, there must exist $N_2 \geq N_1$ such that D meets the circles $\gamma_n = \{z : |z| = R_n\}$, $\gamma_n' = \{z : |z| = (R_n)^{mT}\}$ and $\gamma_n'' = \{z : |z| = \rho_n\}$ for all $n \geq N_2$.

We choose a value $N \in \mathbb{N}$ such that $N \geq N_2$ and note that D must contain a path Γ joining a point $w_N \in \gamma_N$ to a point $w'_{N+1} \in \gamma'_{N+1}$. It is clear that Γ must contain a point $w''_{N+1} \in \gamma''_{N+1}$. Therefore $f(D)$ is a component of $\mathcal{F}(f)$ containing the path $f(\Gamma)$. We know that $M(R_N, f) = R_{N+1}$ and so $|f(w_N)| \leq R_{N+1}$. Also, $L(\rho_{N+1}, f) > (R_{N+2})^{mT}$ and so $|f(w''_{N+1})| > (R_{N+2})^{mT}$. Hence $f(\Gamma)$ must contain an arc joining a point $w_{N+1} \in \gamma_{N+1}$ to a point $w'_{N+2} \in \gamma'_{N+2}$.

We repeat the process inductively to find that $f^k(D)$ is a component of $\mathcal{F}(f)$ containing an arc of $f^k(\Gamma)$ which joins a point $w_{N+k} \in \gamma_{N+k}$ to a point $w'_{N+k+1} \in \gamma'_{N+k+1}$.

Thus, on Γ, the function f^k takes a value of modulus at least R_{N+k}. Since $R_{N+k} \to \infty$ as $k \to \infty$ and D is a component of $\mathcal{F}(f)$, we conclude that $f^k \to \infty$ as $k \to \infty$, locally uniformly in D. It follows that there exists $K \in \mathbb{N}$ such that, for all $k > K$ and all $z \in \Gamma$, we have $|f^k(z)| > 1$.

Thus the family $\{f^k\}_{k>K}$ satisfies the conditions of Lemma 2 on Γ and so there exist constants B and C such that

$$|f^k(z')| < B|f^k(z)|^C \tag{2.6}$$

for all z, z' in Γ, $k > K$.

We know that, for any $k > K$, we can choose $z_k, z'_k \in \Gamma$ such that $f^k(z_k) = w_{N+k} \in \gamma_{N+k}$ and $f^k(z'_k) = w'_{N+k+1} \in \gamma'_{N+k+1}$ and so it follows from (2.6) that

$$M(R_{N+k}, f) = R_{N+k+1} < (R_{N+k+1})^{mT} < B(R_{N+k})^C$$

for each $k > K$. This contradicts the fact that f is a transcendental function, since $R_{N+k} \to \infty$ as $k \to \infty$. This completes the proof of Theorem 2. ∎

Proof of Theorem 1: We only need to show that f satisfies (2.1) when f has positive lower order. In fact for any given $R_1 > 1$, let $T > \rho/\mu$ and $R_{n+1} = M(R_n, f)$, $n = 1, 2, \ldots$. Then (2.1) holds. Otherwise there exists a sub-sequence $\{R_{n_j}\}_{j\geq 1}$ of $\{R_n\}$, $R_{n_j} \to \infty$, as $j \to \infty$, such that

$$M(R_{n_j}^T, f) \leq M(R_{n_j}, f)^{mT}$$

for $j \geq 1$. Therefore

$$\frac{\log\log M(R_{n_j}^T, f)}{\log R_{n_j}} \leq \frac{\log mT + \log\log M(R_{n_j}, f)}{\log R_{n_j}}.$$

By letting $j \to \infty$, we have

$$T\mu \leq \rho.$$

It contradicts the assumption and so Theorem 1 is proved. ∎

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